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THE CRITICAL CASE OF A PAIR OF ZERO ROOTS IN A TWO-DEGREE-OF-FREEDOM HAMILTONIAN SYSTEM†

A. P. MARKEYEV

Moscow

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The problem of the motion of an autonomous two-degree-of-freedom Hamiltonian system in the neighbourhood of its equilibrium position is considered. It is assumed that the characteristic equation of the linearized system has a pair of pure imaginary roots. The roots of the other pair are assumed to be close to or equal to zero, and in the latter case non-simple elementary dividers correspond to these roots. The problem of the existence, bifurcations and orbital stability of families of periodic motions, generated from the equilibrium position, is solved. Conditionally periodic motions are analysed. The problem of the boundedness of the trajectories of the system in the neighbourhood of the equilibrium position in the case when it is Lyapunov unstable, is considered. Non-linear oscillations of an artificial satellite in the region of its steady rotation around the normal to the orbit plane are investigated as an application. © 1998 Elsevier Science Ltd. All rights reserved.

1. FORMULATION OF THE PROBLEM

Suppose the motion of a two-degree-of-freedom system is described by the canonical equations

$$dq_j/dt = \partial H/\partial p_j, \quad dp_j/dt = -\partial H/\partial q_j \quad (j=1,2) \tag{1.1}$$

We will assume that the origin of coordinates $q_j = p_j = 0$ $(j = 1, 2)$ of phase space is an equilibrium position, while the Hamilton function is independent of t and is analytic in a certain neighbourhood of the point $q_j = p_j = 0$ $(j = 1, 2)$.

Suppose the characteristic equation of the linearized system (1.1) has a pair of pure imaginary roots $\pm i\Omega$ $(\Omega > 0)$. We will assume that the roots of the other pair are close to zero and denote their moduli by $\varepsilon|\varkappa|$, where ε is a small parameter $(0 < \varepsilon \ll 1)$. If the quantity $\varkappa$ is equal to zero, we have exact resonance (one of the frequencies of small oscillations is equal to zero). In this case we assume that non-simple elementary dividers correspond to the zero roots of the characteristic equation. When $\varkappa \neq 0$ we have inexact resonance.

If the resonance is exact, the variables q_j and p_j can be chosen, using a normalizing transformation (see [1]), so that in the expansion of the Hamilton function in series, terms higher than the second degree will depend only on q_2 and combinations $q_1^2 + p_1^2$. Below we will consider the case when there are no terms of the third degree in the normal form of the Hamilton function. This case is not unusual, since often, for example, the expansion of the Hamilton function in series contains no forms of odd powers of q_j and p_j.

In the case of inexact resonance the normal form of the Hamilton function will contain the term $\frac{1}{2}\varepsilon\varkappa q_2^2$.

We will assume that normalisation is carried out and, consequently, the Hamilton function in Eqs (1.1) has the following (normal) form

$$\begin{aligned} H = {} & \tfrac{1}{2}\delta_1\Omega(q_1^2+p_1^2) + \tfrac{1}{2}\delta_2 p_2^2 + \tfrac{1}{2}\varepsilon\varkappa q_2^2 + \\ & + \gamma q_2^4 + \tfrac{1}{2}\delta(q_1^2+p_1^2)q_2^2 + \tfrac{1}{4}\sigma(q_1^2+p_1^2)^2 + O_5 \end{aligned} \tag{1.2}$$

In (1.2) the quantities δ_1 and δ_2 are equal to 1 or –1, γ, δ and σ are constants, and O_5 is a converging series, which begins with terms of no less than the fifth power in q_j, p_j $(j = 1, 2)$. Everywhere henceforth we will assume that the coefficient γ in normal form (1.2) is non-zero.

As shown in [1], in the case of exact resonance when the inequality $\delta_2\gamma > 0$ is satisfied, the equilibrium position $q_j = p_j = 0 (j = 1, 2)$ of system (1.1) is stable, and when $\delta_2\gamma < 0$ it is unstable.

One of the aims of this paper is to solve the problem of the existence, bifurcations and orbital stability of the periodic motions which originate from an equilibrium position. Moreover, we investigate in detail

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the nature of the non-linear oscillations in the neighbourhood of the equilibrium position both for exact and for inexact resonance.

Note that Lyapunov's theorem on the holomorphic integral [2] is not applicable to the problem considered in this paper of periodic motions originating from an equilibrium position when there are small oscillations of zero frequency. Other resonance cases, when Lyapunov's theorem is inapplicable, but the characteristic equation has no zero roots, were investigated earlier (see, for example, [3–6], in which an extensive bibliography is given).

2. A HIGHER HAMILTONIAN TRANSFORMATION. PRELIMINARY ANALYSIS OF THE APPROXIMATE SYSTEM

If we drop terms higher than the fourth power in Hamilton function (1.2), the equations of motion will have an integral $r = r_0$, where $2r = q_1^2 + p_1^2$, $r_0 \geqslant 0$ is a constant. We will use the quantity r_0 as the initial value of the variable r in the complete system of the equations of motion, when the terms of all powers are taken into account in the expansion of the Hamilton function in series, and we put $r_0 = \varepsilon|\gamma|^{-1}\rho_0$.

Instead of the variables q_j, p_j $(j = 1, 2)$ we will introduce new canonically conjugate variables φ, ψ, ξ, η using a canonical transformation of the form

$$q_1 = \delta_2\sqrt{2r}\sin\varphi, \quad p_1 = \sqrt{2r}\cos\varphi \; (r = r_0 + \varepsilon^{3/2}|\gamma|^{-1}\psi)$$
$$q_2 = \delta_2\varepsilon^{1/2}|\gamma|^{-1/2}\xi, \quad p_2 = \varepsilon|\gamma|^{-1/2}\eta \tag{2.1}$$

The following Hamilton function corresponds to the equations of motion in the new variables

$$H = \sigma_1\Omega\psi + \varepsilon^{1/2}(\tfrac{1}{2}\eta^2 + \tfrac{1}{2}\sigma_2\nu^2\xi^2 + \sigma_3\xi^4) + O(\varepsilon) \tag{2.2}$$
$$\nu^2 = |\chi|, \; (\nu \geqslant 0), \quad \chi = \varkappa + 2\delta|\gamma|^{-1}\rho_0$$
$$\sigma_1 = \delta_1\delta_2, \quad \sigma_2 = \delta_2 \operatorname{sign}\chi, \quad \sigma_3 = \delta_2 \operatorname{sign}\gamma$$

If we neglect quantities of the order of ε and higher in Hamilton function (2.2), we arrive at an approximate system, which is a set of two oscillators: a linear oscillator with Hamiltonian $\sigma_1\Omega\psi$ and a non-linear one with Hamiltonian which is a set of terms of order $\varepsilon^{1/2}$ from (2.2). This Hamiltonian contains the constant ρ_0 as a parameter.

For a linear oscillator, using (2.1) we have $\psi(t) \equiv 0$, $\varphi(t) = \sigma_1\Omega t + \varphi(0)$. The phase diagrams of the non-linear oscillator for the case when $\nu = 0$ are represented in the ξ, η plane in Fig. 1(b) and (e), where $\sigma_3 = 1$ and $\sigma_3 = -1$, respectively. These diagrams, in particular, illustrate the stability and instability of the origin of coordinates $q_j = p_j = 0$ $(j = 1, 2)$ of initial system (1.1) for exact resonance when $\delta_2\gamma > 0$ and $\delta_2\gamma < 0$, respectively (here, in the expression for $\varkappa$, we have $\varkappa = 0$, $\rho_0 = 0$).

Suppose now that $\nu \neq 0$. In this case it is convenient to change from the variables φ, ψ, ξ, η to canonically conjugate variables w_1, I_1, q, p using the formulae

$$\xi = \nu q, \quad \eta = \nu^2 p, \quad \psi = \nu^3 I_1, \quad \varphi = w_1$$

and also introduce the new time $\tau = \nu t$.

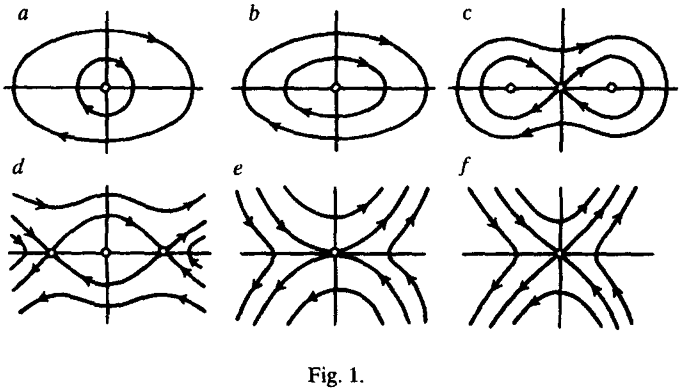

Fig. 1.

In the new variables we have

$$H = \sigma_1\Omega_1 I_1 + \varepsilon^{1/2}(\tfrac{1}{2}p^2 + \tfrac{1}{2}\sigma_2 q^2 + \sigma_3 q^4) + O(\varepsilon) \quad (\Omega_1 = \nu^{-1}\Omega) \tag{2.3}$$

The Hamilton function of the non-linear oscillator of the approximate system now has the form

$$F = \varepsilon^{1/2}(\tfrac{1}{2}p^2 + \tfrac{1}{2}\sigma_2 q^2 + \sigma_3 q^4) \tag{2.4}$$

while the corresponding equations of motion become

$$dq/d\tau = \varepsilon^{1/2}p, \quad dp/d\tau = -\varepsilon^{1/2}(\sigma_2 q + 4\sigma_3 q^3) \tag{2.5}$$

Consider the equilibrium position $q = q_0, p = 0$ of system (2.5). There is always an equilibrium position in which $q_0 = 0$. This equilibrium is stable if $\sigma_2 > 0$ and unstable if $\sigma_2 < 0$.

If $\sigma_2\sigma_3 < 0$, then, in addition to the equilibrium $q_0 = 0, p = 0$, there are two equilibrium in which $p = 0$ while $q_0 = 1/2$ or $-1/2$. These equilibria are stable if $\sigma_2 < 0$ and unstable if $\sigma_2 > 0$.

The phase diagrams of system (2.5) are represented in Fig. 1(a)–(f) in the q, p plane. In Fig. 1(a) and (b) the quantity $\sigma_3 = 1$, where $\sigma_2 = 1$ and -1, respectively. In Fig. 1(d) and (f) $\sigma_3 = -1$ while the value of σ_2 is again equal to 1 and -1 respectively. The centre-type singular points correspond to stable equilibria in Fig. 1(a), (c), (d) and (f) while the saddle singular points correspond to unstable equilibria.

3. FAMILIES OF PERIODIC MOTIONS

The periodic motions of the non-linear oscillator of the approximate system correspond to equilibrium positions. These periodic motions are described by the following formulae in the initial variables q_j, p_j

$$q_1 = \delta_2 |\gamma|^{-1/2}\sqrt{2\varepsilon\rho_0}\sin\varphi, \quad p_1 = |\gamma|^{-1/2}\sqrt{2\varepsilon\rho_0}\cos\varphi(\varphi = \sigma_1\Omega t + \varphi(0)) \tag{3.1}$$

$$q_2 = \delta_2 |\gamma|^{-1/2}\nu\sqrt{\varepsilon}q_0, \quad p_2 = 0$$

Here either $q_0 = 0$ or (when $\sigma_2\sigma_3 < 0$) q_0 can take one of three values: 0, 1/2, $-1/2$.

We will now consider the complete system with Hamilton function (2.3). We will assume that $\nu \neq 0$. We will show that single-parametric families of periodic motions, analytic in $\varepsilon^{1/2}$, exist in the complete system, and approach motions (3.1) as $\varepsilon \to 0$. When $\sigma_2\sigma_3 > 0$ there is a single family of periodic motions and when $\sigma_2\sigma_3 < 0$ there are three. The constant of the energy integral or, which is the same thing, the quantity ρ_0, serves as the parameter of the families.

For the proof we consider the isoenergy level $H = c_* =$ const. Taking expression (2.3) for H into account, and the fact that $I_1(0) = 0$, we expand the equation $H = c_*$ in I_1

$$I_1 = -K = -\sigma_1\Omega_1^{-1}F + O(\varepsilon) \tag{3.2}$$

where F is function (2.4). The terms in (3.2) that are independent of q, p and w_1 are dropped, and $O(\varepsilon)$ is a function that is 2π-periodic in w_1 and analytic in q, p and ε.

On the isoenergy level $H = c_*$ the equations have Hamilton form (the Whittaker equations [7]), and the function K from (3.2) plays the role of the Hamilton function, while the quantity w_1 plays the role of the independent variable. These equations have the form

$$\begin{aligned} dq/dw_1 &= \varepsilon^{1/2}\sigma_1\Omega_1^{-1}p + O(\varepsilon) \\ dp/dw_1 &= -\varepsilon^{1/2}\sigma_1\Omega_1^{-1}(\sigma_2 q + 4\sigma_3 q^3) + O(\varepsilon) \end{aligned} \tag{3.3}$$

We will find a solution of system (3.3) that is 2π-periodic in w_1. We will seek it in the form of the series

$$q = q_0 + \varepsilon^{1/2}q^{(1)} + \varepsilon q^{(2)} + \ldots, \quad p = \varepsilon^{1/2}p^{(1)} + \varepsilon p^{(2)} + \ldots \tag{3.4}$$

Substituting the series into (3.3) and equating terms of like powers of $\varepsilon^{k/2}$, we obtain differential equations for the coefficients $q^{(k)}, p^{(k)}$. For $q^{(1)}, p^{(1)}$ we have the equations

$$dq^{(1)}/dw_1 = 0, \quad dp^{(1)}/dw_1 = 0$$

The general solution of these equations is $q^{(1)} = c^{(1)}, p^{(1)} = d^{(1)}$. The constants $c^{(1)}, d^{(1)}$ are determined at the next step when finding the 2π-periodic solution of the equations for $q^{(2)}, p^{(2)}$

$$\begin{aligned} dq^{(2)}/dw_1 &= \sigma_1\Omega_1^{-1}d^{(1)} + Q^{(2)}(w_1) \\ dp^{(2)}/dw_1 &= -\sigma_1\Omega_1^{-1}(\sigma_2 + 12\sigma_3 q_0^2)c^{(1)} + P^{(2)}(w_1) \end{aligned} \qquad (3.5)$$

where $Q^{(2)}, P^{(2)}$ are 2π-periodic functions not containing $c^{(1)}, d^{(1)}$.

For the periodicity of $q^{(2)}, p^{(2)}$ it is necessary and sufficient to choose $c^{(1)}, d^{(1)}$ so that there are no constant terms in the expansions of the right-hand sides of Eqs (3.5) in Fourier series. The quantity $\sigma_2 + 12\sigma_3 q_0^2$ from the right-hand side of the second of Eqs (3.5) is equal to σ_2 when $q_0 = 0$ and $-2\sigma_2$ when $q_0 = \pm 1/2$. Hence, in view of the assumption $\nu \neq 0$ and the notation from (2.2) it follows that this choice of $c^{(1)}, d^{(1)}$ is always possible. Hence, we obtain from (3.5) that $q^{(2)} = f^{(2)} + c^{(2)}, p^{(2)} = g^{(2)} + d^{(2)}$, where $f^{(2)}$ and $g^{(2)}$ are 2π-periodic in w_1 while $c^{(2)}, d^{(2)}$ are constants determined at the next step.

The process can be continued. For sufficiently small ε, series (3.4) converge [8] and are 2π-periodic functions of w_1, containing ρ_0 as the parameter.

The dependence of w_1 on τ is found from the equation $dw_1/d\tau = \partial H/\partial I_1 = \sigma_1\Omega_1 + O(\varepsilon)$. The period of the solutions with respect to t is equal to the time interval during which w_1 increases by 2π. It tends to $2\pi/\Omega$ as $\varepsilon \to 0$.

Periodic motions from these families, for which the singular saddle points in Fig. 1(a)–(f) correspond as $\varepsilon \to 0$ are orbitally unstable for fairly small ε; this follows from the continuity of the characteristic exponents in ε. As will follow from that is said below (see Section 4), periodic motions from the families to which centre-type singular points correspond as $\varepsilon \to 0$ in Fig. 1(a)–(f) for sufficiently small ε, are orbitally stable.

The conclusions reached give a clear picture of the bifurcations of the periodic motions in the neighbourhood of the origin of coordinates $q_j = p_j = 0$ ($j = 1, 2$) of initial system (1.1) as a function of the coefficients of the normalized Hamiltonian (1.2).

We will first consider the case of exact resonance ($\varkappa = 0$). If $\delta_2\gamma > 0$ (the origin of coordinates is stable), then, when $\delta_2\delta > 0$, a single family of periodic motions is produced from the origin of coordinates, which are orbitally stable; these motions depend on r_0 as on the parameter and as $r_0 \to 0$ transfer into an equilibrium position $q_j = p_j = 0$ ($j = 1, 2$) (Fig. 1a). If $\delta_2\gamma > 0$, while $\delta_2\delta < 0$, three families of periodic motions are produced from the origin of coordinates: one is orbitally unstable while two are orbitally stable: these families "collapse" into the origin of coordinates when $r_0 \to 0$ (Fig. c).

If $\delta_2\gamma < 0$ (the origin of coordinates is unstable), while $\delta_2\delta > 0$, three families of periodic motions are also generated from the origin of coordinates, but now one is orbitally stable while two are unstable (Fig. 1d). If, when $\delta_2\gamma < 0$ we have $\delta_2\delta < 0$, one family of orbitally unstable periodic motions is produced from the unstable origin of coordinates (Fig. 1f).

Suppose now that the resonance is not exact ($\varkappa \neq 0$). A clear representation of the families of periodic motions, generated from the origin of coordinates $q_j = p_j = 0$ ($j = 1, 2$), can again be obtained from Fig. 1(a)–(f). When $\delta_2\gamma > 0$ Fig. 1(a) will be the illustration (the case $\delta_2\chi < 0$, when a single family of orbitally stable motions is generated), and Fig. 1(c) (the case $\delta_2\chi < 0$, when a single family of unstable motions and two families of stable motions are generated). The evolution of the bifurcation pattern when ρ_0 changes in the case of equal signs of the quantities $\varkappa$ and δ is of interest. Suppose, for example, $\delta_2 > 0$ while $\varkappa < 0$, $\delta > 0$. Then, the orbitally stable family of periodic motions, uniquely generated from the origin of coordinates $q_j = p_j = 0$ ($j = 1, 2$) loses stability as ρ_0 increases, and at the instant when stability is lost (when $\rho_0 = \varkappa|\gamma||2\delta|^{-1}$) two orbitally stable families branch off from it (Fig. 1a changes into Fig. 1c). If $\delta_2 > 0$ while $\varkappa < 0$, $\delta < 0$, then, conversely, as ρ_0 increases the unstable family of periodic motions become stable, while the two stable families disappear (Fig. 1c becomes Fig. 1a). The case $\delta_2 < 0$ can be considered similarly.

The case $\delta_2\gamma < 0$ is illustrated in Fig. 1(d) ($\delta_2\chi > 0$; three families of periodic motions are generated: one is stable and two are unstable) and Fig. 1(f) ($\delta_2\chi < 0$; one unstable family is generated). Here, in the case of different signs on the quantities $\varkappa$ and δ, such an evolution of the bifurcation pattern will occur as ρ_0 changes: when $\delta_2 > 0$, $\varkappa > 0$, $\delta < 0$ the unstable families will disappear as ρ_0 increases, while the unstable families will become stable (Fig. 1d becomes Fig. 1f); when $\delta_2 > 0$ and $\varkappa > 0$, $\delta > 0$ the pattern is the opposite: the unstable family becomes stable and two unstable families branch off from it (Fig. 1f becomes Fig. 1d). The case $\delta_2 < 0$ is considered similarly.

4. CONDITIONALLY PERIODIC MOTIONS

We will consider in more detail the nature of the non-linear oscillations in the neighbourhood of an equilibrium position $q_j = p_j = 0$ ($j = 1, 2$) of system (1.1). Confining ourselves to the case when $\nu \neq 0$, we will investigate the transformed equations of motion with Hamilton function (2.3).

We will first consider the approximate system. In this $I_1(\tau) \equiv 0$, and the variables q and p satisfy Eqs (2.5). These equations have the integral $f = h =$ const, where F is function (2.4). The phase patterns are shown in Fig. 1(a)–(f). Below we consider four cases, when there are regions on the phase patterns filled with closed trajectories. The standard notation is used: k is the modulus of the elliptic functions and integrals, and K and E are the complete elliptic integrals of the first and second kind, respectively. For convenience we will also introduce the following notation

$$h = \varepsilon^{1/2} a/16,\quad b = \sqrt{1+a},\quad c = \sqrt{1-a},\quad e = (b-1)^{1/2}/2$$

$$c_1 = (1-b)^{1/2}/2,\quad c_2 = (1+b)^{1/2}/2,\quad e_1 = (1-c)^{1/2}/2,\quad e_2 = (1+c)^{1/2}/2$$

1. *The case $\sigma_3 = 1$, $\sigma_2 = 1$ (Fig. 1a).* Here $a \geqslant 0$. When $a = 0$ we have the equilibrium $q = p = 0$ of system (2.5). The case $a > 0$ corresponds to oscillations in the neighbourhood of this equilibrium. Here $-e \leqslant q(\tau) \leqslant e$. If we assume that $q(0) = -e$, we have

$$q = -e\,\mathrm{cn}(\sqrt{\varepsilon b}\tau, k),\quad k^2 = (1-b^{-1})/2$$

The oscillation frequency (with respect to τ) is calculated from formula $\omega = \pi\sqrt{(\varepsilon b)}K^{-1}/2$.

When $a > 0$, we can introduce the "action-angle" variables I_2 and w_2 [9]. Calculations show that

$$I_2(a) = (6\pi)^{-1} b^{1/2}[(b+1)K - 2E] \tag{4.1}$$

Inverting the function (4.1), taking the above notation for a and b into account, we obtain $h(I_2) = \varepsilon^{1/2}H^{(1)}(I_2)$, where $\omega = \varepsilon^{1/2}\partial H^{(1)}/\partial I_2$. In a small neighbourhood of the point $q = p = 0$ (where $0 < a \ll 1$) the following expansions hold

$$\omega = \varepsilon^{1/2}(1 + \frac{3}{16}a + \ldots),\quad h = \varepsilon^{1/2}(I_2 + \frac{3}{2}I_2^2 + \ldots)$$

Everywhere in the region of oscillations ($a > 0$) the function $h(I_2)$ satisfies the non-degeneracy condition $d^2h/dI_2^2 \neq 0$. In fact, calculations show that

$$\frac{d^2H^{(1)}}{dI_2^2} = \frac{6\pi^3 I_2}{a\sqrt{b}K^3} > 0$$

Hence, the non-degeneracy of h follows from the equality $h = \varepsilon^{1/2}H^{(1)}$.

2. *The case when $\sigma_3 = 1$, $\sigma_2 = -1$, $-1 < a < 0$ (the region of oscillations in Fig. 1c).* When $\sigma_3 = 1$, $\sigma_2 = -1$ the quantity a satisfies the inequality $a \geqslant -1$. If $a = -1$, we have the equilibrium $q = 1/2$, $p = 0$ or $q = -1/2, p = 0$. When $-1 < a < 0$ oscillations occur in the neighbourhood of these equilibria. We will consider them in more detail, confining ourselves, for example, to the case of oscillations in the neighbourhood of the equilibrium $q = 1/2, p = 0$. Then $c_1 \leqslant q(\tau) \leqslant c_2$, where

$$q = c_2\,\mathrm{dn}(\sqrt{2\varepsilon}c_2\tau, k),\quad k^2 = 2b(1+b)^{-1}$$

The oscillation frequency $\omega = \pi\sqrt{(2\varepsilon)}c_2K^{-1}$. The action variable is calculated from the formula

$$I_2 = \sqrt{2}(6\pi)^{-1}c_2(E - 4c_1^2K)$$

In a small neighbourhood of the equilibrium we have

$$\omega = \sqrt{2\varepsilon}(1 - \frac{3}{16}(a+1) + \ldots),\quad h = \sqrt{\varepsilon}(-\frac{1}{16} + \sqrt{2}I_2 - 3I_2^2 + \ldots)$$

Everywhere in the region $-1 < a < 0$ the function $h(I_2) = \varepsilon^{1/2}H^{(1)}(I_2)$ satisfies the non-degeneracy condition, since

$$\frac{d^2H^{(1)}}{dI_2^2} = -\frac{6\sqrt{2}\pi^3 c_2 I_2}{c_1^2(a+1)K^3} < 0$$

The case $a = 0$ corresponds either to the equilibrium $q = p = 0$ or to trajectories doubly asymptotic to it—separatrices which, in Fig. 1(c), distinguish regions of oscillations in the neighbourhood of the equilibria $q = \pm 1/2, p = 0$ from regions of rotation, in which the trajectories encompass all three equilibria. In the region of oscillations, the frequency ω decreases as one approaches the separatrice, and in sufficient proximity to the separatrice, where $0 < -a \ll 1$, $\omega \cong -2\pi\varepsilon^{1/2}\ln^{-1}(-a)$.

3. *The case $\sigma_3 = 1$, $\sigma_2 = -1$, $a > 0$ (the region of rotations in Fig. 1c).* Here

$$q = -c_2\,\mathrm{cn}(\sqrt{\varepsilon b}\tau, k), \quad k^2 = \tfrac{1}{2}(1 + b^{-1})$$

$$\omega = \pi\sqrt{\varepsilon b}(2K)^{-1}, \quad I_2 = (3\pi)^{-1} b^{1/2}(2e^2K + E)$$

$$\frac{d^2H^{(1)}}{dI_2^2} = \frac{6\pi^3 I_2}{a\sqrt{b}K^3} > 0$$

In the region of the separatrice, where $0 < a \ll 1$, we have $\omega \cong -\pi\varepsilon^{1/2}\ln^{-1}a$.

4. *The case $\sigma_3 = -1$, $\sigma_2 = 1$, $0 < a < 1$ (the region of oscillations in Fig. 1d).* When $\sigma_3 = -1$, $\sigma_2 = 1$ and $a = 0$ we have the equilibrium $q = p = 0$, and when $a = 1$ either the equilibrium $q = \pm 1/2, p = 0$ or a doubly asymptotic trajectory connecting them—a separatrice. When $0 < a \leqslant 1$ we have a region of oscillations in the neighbourhood of the equilibrium $q = p = 0$. In the region of oscillations

$$q = e_1\,\mathrm{sn}(\sqrt{2\varepsilon}e_2\tau, k), \quad k^2 = (1 - c)(1 + c)^{-1}$$

$$\omega = \pi\sqrt{2\varepsilon}e_2(2K)^{-1}, \quad I_2 = (3\pi)^{-1}\sqrt{2}e_2(E - cK)$$

$$\frac{d^2H^{(1)}}{dI_2^2} = -\frac{3\sqrt{2}\pi^3 e_2 I_2}{4e_1^2(1-a)K^3} < 0$$

In the region of the equilibrium $q = p = 0$ the following expansions hold

$$\omega = \varepsilon^{1/2}\left(1 - \frac{3}{16}a + \ldots\right), \quad h = \varepsilon^{1/2}\left(I_2 - \frac{3}{2}I_2^2 + \ldots\right)$$

while in the region of the separatrice $\omega \cong -\pi\sqrt{(2\varepsilon)}\ln^{-1}(1 - a)$.

We will now consider the complete system with Hamilton function (2.3). For each of the four cases considered in "action-angle" variables I_1, I_2, w_1, w_2, we have

$$H = H^{(0)}(I_1) + \varepsilon^{1/2}H^{(1)}(I_2) + \varepsilon H^{(2)}(I_1, I_2, w_1, w_2; \varepsilon^{1/2}), \quad H^{(0)} = \sigma_1\Omega_1 I_1 \tag{4.2}$$

The function $H^{(1)}(I_2)$ in each of cases 1–4 is defined above. In the regions of oscillations and rotations considered the function (4.2) is 2π-periodic in w_1 and w_2 and analytic with respect to its arguments. Here, we have the case of natural degeneracy [10], since when $\varepsilon = 0$ Hamilton function (4.2) depends on only one of the action variables.

As follows from the above results, Hamiltonian (4.2) satisfies the conditions

$$\frac{\partial H^{(0)}}{\partial I_1} \neq 0, \quad \frac{\partial H^{(1)}}{\partial I_2} \neq 0, \quad \frac{\partial^2 H^{(1)}}{\partial I_2^2} \neq 0$$

Hence it follows from [10, 11] that in the complete system (in the regions considered in cases 1–4) the motion for the majority of the initial conditions will be conditionally periodic with frequencies (with respect to t) Ω and ν, ω; only a fraction $O(\exp(-d_1\varepsilon^{-1/2}))$, $d_1 > 0$ – const of phase space is not filled with conditionally periodic trajectories.

In this case, for all the initial conditions, the values of $I_j(t)$ ($j = 1, 2$) in the complete system for all t are close to their initial values: $|I_j(t) - I_j(0)| < d_2\varepsilon^{1/2}$ (d_2 = const).

Hence, follows, in particular, the orbital stability of the periodic motions from the families, to which as $\varepsilon \to 0$ centre-type singular points in Fig. 1(a)–(f) correspond (see Section 3).

5. NOTE

Suppose we have exact resonance, i.e. in (1.2) $\varkappa$ is equal to zero. Then [1] when the inequality $\delta_2\gamma < 0$ is satisfied the equilibrium $q_j = p_j = 0$ $(j = 1, 2)$ of system (1.1) is unstable. It turns out, however, that if the coefficient γ and δ of the normal form of the Hamilton function (1.2) have opposite signs, then, despite the instability when $\delta_2\gamma < 0$ of the equilibrium $q_j = p_j = 0$ $(j = 1, 2)$ of system (1.1), its trajectories, beginning fairly close to the equilibrium and such that $r_0 \neq 0$, for all t may remain in as small a neighbourhood of the equilibrium as desired.

In fact, the case $\varkappa = 0$, $\delta_2\gamma < 0$, $\gamma\delta < 0$ is illustrated in Figs 1(d) and (e), where $r_0 \neq 0$ and $r_0 = 0$, respectively. It follows from the results of the previous section that if $\delta_2\gamma < 0$ and $\gamma\delta < 0$ (i.e. $\sigma_3 = -1$, $\sigma_2 = 1$) and at the initial instant $t = 0$ we have $q_1^2(0) + p_1^2(0) = 2r_0 \neq 0$ $(0 < r_0 \ll 1)$, then for all $t > 0$ the value of r is close to r_0 while $q_2(t)$ and $p_2(t)$ are as close as desired to the origin of coordinates if the quantities $|q_2(0)|$, $|p_2(0)|$ are sufficiently small. More exactly, the inequalities $|q_2(t)| < |\delta/(2\gamma)|^{1/2} r_0^{1/2}$, $|p_2(t)| < |\delta/(2\gamma)|^{-1/2} r_0$ will be satisfied for all $t > 0$. These inequalities follow from the estimates of the dimensions of the region of oscillations in Fig. 1(d).

To illustrate the above, consider the following model example. Suppose a heavy point mass moves over a fixed absolutely smooth surface, which differs only slightly from a cylindrical surface with a horizontal generatrix. We will choose the scale so that the point mass, the acceleration due to gravity and the non-zero curvature of the surface at the equilibrium position of the point are equal to unity. We will refer the motion to a fixed system of coordinates xyz, the z axis of which is directed vertically upward. We will specify the surface by the equation

$$z(x,y) = \tfrac{1}{2}x^2 + x^2y^2 - y^4$$

The equations of motion of the point can be written in the form

$$\ddot{x} + (\ddot{z}+1)z_x = 0, \quad \ddot{y} + (\ddot{z}+1)z_y = 0$$

The origin of coordinates $x = y = 0$ is an equilibrium, and this equilibrium is unstable. In fact, one can verify that the equations of motion allow of particular solutions for which $x(t) \equiv 0$, while $y(t)$ satisfies the differential equation

$$(1+16y^6)\ddot{y} + 48y^5\dot{y}^2 - 4y^3 = 0 \tag{5.1}$$

These particular solutions describe the motion of a point mass in the vertical plane yz along an absolutely smooth curve $z = -y^4$. For all the solutions of Eq. (5.1), apart from $y(t) \equiv 0$ we have $|y(t)| \to \infty$ if $t \to \infty$. Hence, it follows that the equilibrium $x = y = 0$ is unstable.

However, the point mass for all t can remain in a small neighbourhood of the equilibrium $x = y = 0$, if its motion begins fairly close to the origin of coordinates with a small initial velocity, where $x^2(0) + \dot{x}^2(0) \neq 0$.

In order to show that this assertion is true, we will obtain the normal form of the Hamilton function. Introducing the generalized momenta p_x and p_y in the usual way, we obtain that in the neighbourhood of an equilibrium

$$H = \tfrac{1}{2}(p_x^2 + p_y^2) + \tfrac{1}{2}x^2 + x^2y^2 - y^4 - \tfrac{1}{2}x^2p_x^2 + O_6$$

where O_6 is a convergent series in powers of x, t, p_x and p_y, which begins with a term no less than the sixth power.

Exact resonance occurs, since one of the frequencies of the small oscillations is equal to zero. Here non-simple elementary dividers correspond to the corresponding zero roots of the characteristic equation. The quadratic part of the Hamilton function (6.3) has a normal form. Fourth-power terms can be normalized using univalent canonical replaclement of the variables $x, y, p_x, p_y \to q_1, q_2, p_1, p_2$, specified by the following generating function

$$S = xp_1 + yp_2 + S_4$$

$$S_4 = \frac{1}{16}(x^3p_1 - 4x^2yp_2 + 8xy^2p_1 - xp_1^3 - 4xp_1p_2^2 + 4yp_1^2p_2)$$

We write the Hamilton function, normalized up to terms of the fourth power inclusive, in the form

$$H = \tfrac{1}{2}(q_1^2 + p_1^2) + \tfrac{1}{2}p_2^2 - q_2^4 + \tfrac{1}{2}(q_1^2 + p_1^2)q_2^2 - 1/16(q_1^2 + p_1^2)^2 + O_6$$

Hence, in (1.2) we have $\delta_2 = 1$, $\gamma = -1$, $\delta = 1$. Taking into account the form of the replacement of variables

$x, y, p_x, p_y \to q_1, q_2, p_1, p_2$, it therefore follows that the above assertion regarding the limited nature of the motion of the point in the neighbourhood of its unstable equilibrium $x = y = 0$ is true.

6. THE PERIODIC MOTIONS OF AN ARTIFICIAL SATELLITE CLOSE TO ITS STEADY ROTATION

We will consider the motion of a dynamically symmetric artificial satellite—a rigid body—about its centre of mass in a circular orbit. Suppose $OXYZ$ is an orbital system of coordinates with origin at the centre of mass of the satellite (the OZ axis is directed along the radius vector of the centre of mass, and OX and OY are directed along the transversal and binormal to the orbit, respectively), while $Oxyz$ is a system of coordinates, rigidly connected to the satellite, formed by the principal central axes of inertia (the Oz axis is directed along the axis of symmetry of the satellite). We will specify the orientation of the connected system of coordinates with respect to the orbital system using the Euler angles ψ, θ, φ. We will denote the corresponding generalized momenta by $p_\psi, p_\theta, p_\varphi$.

Suppose A and C are the equatorial and polar moments of inertia of the satellite while ω_0 is the average motion of the centre of mass in the orbit. The coordinate φ is cyclical, and hence $p_\varphi = C\Omega_0 = \text{const}$, where Ω_0 is the projection of the angular velocity of the satellite onto its axis of symmetry.

We know [12], that for any Ω_0 the equations of motion allow of the solution $\psi = \pi$, $\theta = \pi/2$, $p_\psi = p_\theta = 0$. For this solution the axis of symmetry of the satellite is perpendicular to the orbit plane, while the satellite rotates around the axis of symmetry with an angular velocity Ω_0. The solution of the problem of the stability of this steady rotation of the satellite depends on two dimensionless parameters α, β, ($\alpha = C/A$, $\beta = \theta_0/\omega_0$).

This problem has been investigated in detail (see [12–14] and the bibliography given there). In particular, stability along the curve $\alpha\beta - 1 = 0$ (when $2/3 < \alpha \leqslant 2$) and along the curve $\alpha\beta + 3\alpha - 4 = 0$ (when $0 < \alpha \leqslant 2$) was investigated in [13]. These curves separate regions of stability and instability in the α, β plane; for values of α and β belonging to these curves, the characteristic equation of the linearized equations of perturbed motion has a pair of pure imaginary and a pair of zero roots, to which non-simple elementary dividers correspond. It was shown in [13] that on the curve $\alpha\beta - 1 = 0$ when $2/3 < \alpha < 1$ steady rotation is unstable, while when $1 < \alpha \leqslant 2$ it is stable. On the curve $\alpha\beta + 3\alpha - 4 = 0$ when $0 < \alpha < 1$ and $4/3 < \alpha \leqslant 2$ there is stability, and when $1 < \alpha < 4/3$ there is instability.

Basing ourselves on the results obtained in Sections 2–4, we will consider the periodic motions of the axis of symmetry of the satellite in the neighbourhood of the normal to the orbit plane for values of the parameters α and β lying on the above-mentioned boundaries of the stability regions or close to them. To do this we must obtain the normal form (1.2) of the Hamiltonian of the perturbed motion.

If we introduce the perturbations Q_j and P_j, by putting

$$\theta = \pi/2 + Q_1, \quad \psi = \pi + Q_2, \quad p_\theta = A\omega_0 P_1, \quad p_\psi = A\omega_0 P_2$$

the Hamilton function of the perturbed motion can be written in the form of a series, in which there are no forms of odd powers in Q_j, P_j ($j = 1, 2$).

Suppose $\alpha\beta - 1 = \varepsilon\Delta_1$, $\Delta_1 = \operatorname{sign}(\alpha\beta - 1)$. Then, as calculations show, in normal form (1.2) we have

$$\begin{gathered}\delta_1 = 1, \quad \delta_2 = \operatorname{sign}(\alpha - 1), \quad \Omega = \sqrt{3\alpha - 2} + O(\varepsilon) \\ \varkappa = 3\Delta_1|\alpha - 1|\Omega^{-2} + O(\varepsilon), \quad \gamma = 9/8(\alpha - 1)^2\Omega^{-4} + O(\varepsilon) \\ \delta = -27/4\delta_2(\alpha - 1)^2\Omega^{-5} + O(\varepsilon)\end{gathered} \tag{6.1}$$

If $\alpha\beta + 3\alpha - 4 = \varepsilon\Delta_2$, $\Delta_2 = \operatorname{sign}(\alpha\beta + 3\alpha - 4)$, the coefficients of normal form (1.2) will be

$$\begin{gathered}\delta_1 = 1, \quad \delta_2 = -\operatorname{sign}(\alpha - 1), \quad \Omega = \sqrt{9\alpha^2 - 15\alpha + 7} + O(\varepsilon) \\ \varkappa = 3\Delta_2|\alpha - 1|\Omega^{-2} + O(\varepsilon), \quad \gamma = 9/8(\alpha - 1)^2(4 - 3\alpha)\Omega^{-4} + O(\varepsilon) \\ \delta = 9/4\delta_2(\alpha - 1)^2(72\alpha^2 - 111\alpha + 44)\Omega^{-5} + O(\varepsilon)\end{gathered} \tag{6.2}$$

Expressions (6.1) and (6.2), using the results obtained in Sections 2–4, give a detailed picture which describes the form of the non-linear oscillations of the axis of symmetry of the satellite close to the normal to the orbit plane for values of the parameters α, β, lying in the neighbourhood of curves $\alpha\beta - 1 = 0$ and $(\alpha\beta + 3\alpha - 4)$. For brevity we will only consider the case of exact resonance (i.e. when $\Delta_1 = 0$ and $\Delta_2 = 0$ in (6.1) and (6.2)).

From curve $\alpha\beta - 1 - 0$ and $2/3 < \alpha < 1$ (where the steady rotation is unstable) there is one family of orbitally unstable periodic motions with a period close to $2\pi/\Omega$ (Fig. 1f); if $1 < \alpha \leqslant 2$ (when the steady rotation is stable), two families of orbitally stable periodic motions and one orbitally unstable motion exist (Fig. 1c).

On the curve $\alpha\beta + 3\alpha - 4 = 0$ when $0 < \alpha < 1$ and $4/3 < \alpha \leqslant 2$ (where the steady rotation is stable) there is one family of orbitally stable periodic motions (Fig. 1a). If $1 < \alpha < 4/3$, there are two families of orbitally unstable periodic motions and one family of orbitally stable periodic motions (Fig. 1d).

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